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Abstract

Doodles were introduced in [7] but were restricted to embedded circles in the 2-sphere. Khovanov, [15], extended the idea to immersed circles in the 2-sphere. In this paper we further extend the range of doodles to any closed oriented surfaces. Uniqueness of minimal representatives is proved, and various example of doodles are given with their minimal representatives. We also introduce the notion of virtual doodles, and show that there is a natural one-to-one correspondence between doodles on surfaces and virtual doodles on the plane.

keywords: doodles, virtual doodles, minimal diagrams, immersed circles

Mathematics Subject Classification 2010: 57M25, 57M27

1 Introduction

Doodles were first introduced by the second author and Taylor in [7]. The original definition of a doodle was a collection of embedded circles in the 2-sphere S^2 with no triple or higher intersection points. Khovanov, [15], extended the idea to allow each component to be an immersed circle in S^2. Further references on plane curves are [1] and [2].

In this paper, we further extend the range of doodles to immersed circles in closed oriented surfaces of any genus. Doodles on surfaces can be regarded as equivalence classes of generically immersed circles in closed oriented surfaces, called regular representatives or diagrams, under an equivalence relation. The equivalence relation is generated by introducing/removing locally a monogon or bigon and surgeries on ambient surfaces avoiding the diagrams (Section 2).

A doodle diagram on a surface is called minimal if the interior of every region is simply connected and there are no monogons and bigons. In Section 3, we prove uniqueness of minimal diagrams, which means that for each doodle, there is a unique minimal diagram representing the doodle (Theorem 3.7). It is analogous to Kuperberg's theorem, [18], in virtual knot theory.

To show this theorem, we generalise Newman's simplification procedure, [20]. The main result of Section 3 is Theorem 3.4: The graph $\mathcal{D}$ of doodle diagrams with levels is a 'proof reduction graph'. Theorem 3.4 implies that the uniqueness theorem (Theorem 3.7) and a theorem on characterization of minimal diagrams (Theorem 3.8), which claims that a diagram is minimal if and only if it is a diagram with minimum crossing number and the ambient surface has minimum genus and maximum component number.

In Section 4, we observe minimal doodle diagrams and provide examples of planar doodles by giving sequences of minimal planar diagrams.

The notion of a virtual doodle is introduced in Section 5, and it is proved in Section 6 that there is a natural one-to-one correspondence between virtual doodles and doodles on surfaces (Theorem 6.4). This correspondence is analogous to the correspondence between virtual links and stable equivalence classes of link diagrams on surfaces in [5, 14].

Section 7 is devoted to showing examples of minimal virtual doodle diagrams and an observation on a relationship between the genera of doodles and virtual crossings of virtual doodles.

It is shown in [9] that planar doodles induce commutator identities in the free group, [12]. Commutator identities related to doodles on surfaces will be discussed in a later paper. For a unified treatment of generalized knot theories see [11].

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2 Definitions

A **doodle** is represented by a map $f : \coprod_i S^1_i \to \Sigma$ from n disjoint circles to a closed oriented surface Σ so that $|f^{-1}f(x)| < 3$ for all $x \in \coprod_i S^1_i$. That is, no triple or higher multiple points are created. To avoid "wild"doodles, we further assume that f is a smooth map with a (topological) normal bundle.

The map f restricted to any one circle is called a **component**.

Two representatives are said to be **equivalent** if they are equivalent under the equivalence generated by 1. 2. and 3. defined as follows.

1. Homeomorphic equivalence,

2. Homotopy through doodle representatives,

3. Surface surgery disjoint from the diagram.

The equivalence class is called a **doodle**, or a **doodle on a surface**. However, as is the usual custom, we will often not distinguish between a doodle and its representative.

A doodle representative is **regular** if it is an immersion whose multiple points are a finite number of transverse crossings. Clearly any doodle can be regularly represented. The image of a regular representative is called a **diagram** on the surface. A pair (Σ, D) of a diagram D on a surface Σ is also called a diagram on a surface.

A representative $f : \coprod_i S^1_i \to \Sigma$ or a diagram is called **admissible** if each connected component of the surface Σ intersects with the image of f. Throughout this paper we always assume that representatives and diagrams are admissible unless otherwise stated.

A doodle is called **planar** if there is a representative on the 2-sphere. Although such a representative is often depicted on the plane, one should consider it on the 2-sphere.

Figure 1 shows diagrams of two planar doodles. The first, called the **poppy**, has one component and the other with 3 components is called the **Borromean** doodle.

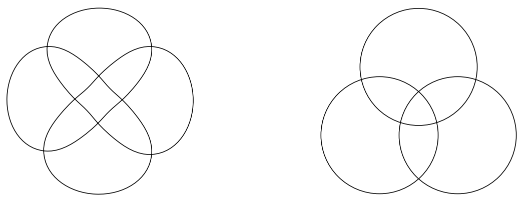

Figure 1: The Poppy and the Borromean Doodle

These are the first members of an infinite family of planar doodles considered later in the paper.

The **Hopf doodle** is represented by the longitude and meridian of a torus. That is $S^1 \times * \cup * \times S^1 \subset S^1 \times S^1$.

1. Homeomorphic equivalence means that the doodles are "topologically the same". So for two doodles f and g with the same number of components there are homeomorphisms $s : \coprod_i S^1_i :\to \coprod_i S^1_i$ and $t : \Sigma \to \Sigma$ which make the square

$$\begin{array}{ccc} \coprod_i S^1_i & \xrightarrow{f} & \Sigma \\ \downarrow s & & t \downarrow \\ \coprod_i S^1_i & \xrightarrow{g} & \Sigma \end{array}$$

commute. Here we assume that t respects the orientation of Σ. When we consider **oriented** (or **unoriented**) doodles, it is (or is not) required that the homeomorphism $s : \coprod_i S^1_i :\to \coprod_i S^1_i$ respects the standard orientation of the circles. When we consider **ordered** (or **unordered**) doodles, it is (or is not) required that the homeomorphism $s : \coprod_i S^1_i :\to \coprod_i S^1_i$ respects the indices.

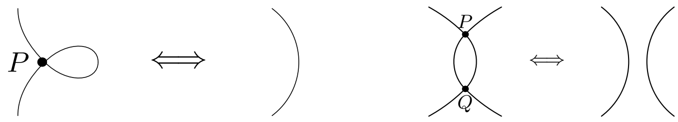

Figure 2: $H_1^{\pm 1}$ and $H_2^{\pm 1}$

2. Any homotopy of regular doodles can be assumed to be divided up into Reidemeister type moves: H_1^+ which generates a curl, H_1^- which deletes a curl, H_2^+ which generates a bigon and H_2^- which deletes it. See Figure 2.

Under H_1^+ a self crossing point P and a monogon with empty interior are introduced and under H_1^- both are eliminated.

Under H_2^+ two crossing point P, Q are introduced and a bigon with empty interior between them. The crossing points may or may not be crossings of different components. Under H_2^- the crossing points and the bigon are eliminated.

3. Surface surgery can be divided into a sequence of handle additions, h^+, and handle eliminations, h^-. These operations have to be disjoint from the doodle. Handle addition can be described as follows. Let D_0, D_1 be disjoint closed discs in the surface disjoint from the diagram. Remove the interiors of the discs and add an annulus, $A = S^1 \times [0,1]$, by its boundary to the boundary of the discs. The circle $b = S^1 \times 1/2$ is called the **belt** of the handle. See Figure 3.

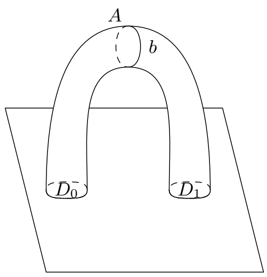

Figure 3: A Handle

The elimination of a handle is the reverse of this procedure. Let b, (the belt of a handle), be a simple closed curve in the surface disjoint from the doodle. Then a regular neighbourhood, A of b is homeomorphic to an annulus and can be chosen disjoint from the doodle. Remove the interior of the annulus and glue two discs via their boundary circles to the boundary of the annulus. Since we consider admissible diagrams, we avoid handle eliminations which make the diagrams inadmissible.

3 Uniqueness of Minimal Doodle Diagrams

In this section we see that every doodle has a unique representative doodle diagram which is minimal with respect to the number of crossing points and the genus of the underlying surface. To do this we first formalise a procedure which is common in mathematical proofs and was introduced by Newman, [20]. It has since been recently considered by Matveev, [19] and Bergman, [4].

Let G be a graph. We will denote the vertices of G by capital Roman letters and the edges by their end points: for example $e = KL$.

As usual, a **path** in G from A to B is a sequence of vertices

$$A = K_0, K_1, K_2, \cdots, K_{n-1}, K_n = B$$

in which $K_{i-1}K_i$ is an edge, $i = 1, \ldots, n$.

A path is called **simple** if it has no re-entrant vertices, $K_i = K_j$ where $0 < i < j < n$. Any path can be replaced by a simple path with the same end points.

We will call G a **graph with levels** if each vertex, K, of G has a level, $|K|$, which is an element of a totally ordered set, called the **level set**, and that any two end vertices of an edge have different levels. This defines an ordering on the edges of G. If an edge e of G has vertices K and L and $|K| > |L|$ then e is oriented from K to L. We will write an edge oriented in this manner as $e = K \searrow L$ or $L \swarrow K$ and picture K on the page as being above L. We say that K **collapses** to L or L **expands** to K.

A path of the form

$$A \searrow K_1 \searrow K_2 \cdots \searrow K_{n-1} \searrow B$$

is called a **descending** path. The inverse of a descending path is called an **ascending** path.

The first condition we impose on G is the **finite descending path property**.

FDPP: There are no infinite descending paths.

A **root** of G is a sink. That is a vertex, R, with no outgoing edges, $R \searrow L$. So a root is a local minimum.

Lemma 3.1 *If a graph with levels has the FDPP property then every vertex is either a root or is connected to a root by a descending path.*

Proof. If K is not a root then it has a descending edge, $K \searrow K_1$. If K_1 is not a root it has a descending edge, $K_1 \searrow K_2$ and so on. By the FDPP property this process must terminate after a finite number of steps with a root. $\square$

From now on we will assume that all graphs have the FDPP. Having this condition, which guarantees the existence of roots, we now look at conditions which make the root unique.

To this end we consider the **unique root property** and the **diamond condition**.

URP: Every pair of vertices in a path component descend to a unique root.

To define the diamond condition we need a few definitions.

A **peak, (valley)** is an ascending (descending) path composed with a descending (ascending) path. A peak (valley) is **simple** if it consists of just two edges.

DC: Any peak can be replaced by a valley with the same end points. In particular if U descends to X and Y then either $X = Y$ or there is a vertex V which ascends to X and Y.

$$\begin{array}{ccccccc} & & & U & & & \\ & & \swarrow & & \searrow & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ & \swarrow & & & & & \searrow \\ X & & & & & & Y \\ & \searrow & & & & \swarrow & \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & \searrow & & \swarrow & & \\ & & & V & & & \end{array}$$

Lemma 3.2 *URP and DC are equivalent conditions.*

Proof. Suppose a graph has the URP and U descends to X and Y. Then X and Y are clearly in the same path component and so both descend to a common root, V say.

Conversely suppose the graph has the DC and X descends to different roots R_1 and R_2. Then unless $R_1 = R_2$ this contradicts the DC. So every vertex descends to a unique root.

Now suppose that X and Y are joined by a path

$$X = K_0, K_1, K_2, \cdots, K_{n-1}, K_n = Y$$

and yet descend to different roots R_1 and R_2. Then somewhere in the path, vertices K_i and K_{i+1} descend to different roots. We may as well assume that $K_i \searrow K_{i+1}$. Then K_i descends to one root and via K_{i+1} to another, contradicting the above. $\square$

We say that a graph with levels is a **proof reduction** graph if it has both the FDPP and the DC/URP properties. By the above, all path components of a proof reduction graph have a unique root. This is clearly a useful property but we need practical methods to recognise such a graph. We do this by localising the DC as follows.

LDC: A graph has the **local diamond condition** if given a simple peak $X \swarrow U \searrow Y$ there is a path, $X = K_0, K_1, K_2, \cdots, K_{n-1}, K_n = Y$, from X to Y such that if the path contains a simple peak $K_{i-1} \swarrow K_i \searrow K_{i+1}$ then there is an edge $U \searrow K_i$.

Lemma 3.3 *The local diamond condition, LDC and the diamond condition, DC are equivalent.*

Proof. Clearly DC implies LDC because a valley does not have a peak.

Now consider a graph with the LDC. Because of FDPP every vertex which is not a root is connected to at least one root by a descending path. Our task is to show that this root is unique.

Let us call a vertex **regular** if it is connected to a unique root by a descending path: otherwise we call it **irregular**. Clearly regular vertices exist. A root is an example. Our task is to show that irregular vertices do not exist. We will assume the contrary and obtain a contradiction.

In that case there must be an irregular vertex L such that for every edge $L \searrow K$ the vertex K is regular. If not we could construct an infinite descending path of irregular vertices. Indeed, we will chose L so that every descending path from L must consist of regular vertices apart from L.

For such an irregular vertex L there is a simple peak $X \swarrow L \searrow Y$ such that X, Y descend to unique but different roots. Chose X and Y so that the path, $X = K_0, K_1, K_2, \cdots, K_{n-1}, K_n = Y$ predicted by the LDC has shortest possible length.

The hypothesis of the LDC means that if the path joining X to Y has a simple peak $K_{i-1} \swarrow K_i \searrow K_{i+1}$ as a subpath then there is an edge $L \searrow K_i$. This means that K_i is regular and so descends to a unique root.

This root must be different from one of the distinct roots of the pair X, Y. So either the pair X, K_i or the pair K_i, Y has a shorter path.

So the path joining X to Y has no simple peaks and is therefore either a) ascending, b) descending or c) a valley. If a) or b) then X, Y have a common root. If c) and If V is the base of this valley then V has a unique root which must be the same for X and Y.

Therefore irregular vertices cannot exist and all vertices are regular and connected to a unique root by a descending path. Hence the URP is satisfied which implies the LDC$\square$

We now consider examples of proof reduction graphs.

Free Groups The vertices are words in the symbols $x \in X$ and $x^{-1} \in X^{-1}$. The level of a word is its length. The expansions are insertions of pairs xx^{-1} or $x^{-1}x$ in the words. It is easy to see that this has the local diamond condition. Hence every word is equivalent to a unique reduced word.

The singular braid monoid embeds in a group, [10] Here the vertices are singular braids up to braid equivalence. The expansions are the introduction of pairs of cancelling singular crossings. The levels are the number of singular crossings.

The graph, $\mathcal{D}$, we are interested in has vertices consisting of (admissible and ordered) oriented (or unoriented) doodle diagrams on surfaces, $S = (\Sigma, D)$. We assume that homeomorphic diagrams are the same vertex of $\mathcal{D}$. The level of a vertex is the number of crossings minus the Euler characteristic of the surface. The moves $H_1^{\pm 1}, H_2^{\pm 1}$ and $h^{\pm 1}$ raise or lower the level and correspond to the edges of the graph.

Theorem 3.4 *The graph, $\mathcal{D}$, of oriented (or unoriented) doodle diagrams with moves and levels defined above is a proof reduction graph.*

Before proving this theorem, we prepare some terminology and a lemma.

A **trivial doodle diagram with one component** is a doodle diagram such that the diagram is a simple closed curve and the surface is a 2-sphere. A **trivial doodle diagram with n components** for $n \geq 1$ is a doodle diagram which is the disjoint union of n trivial doodle diagrams with one component.

A **floating component** of a doodle diagram is a component which bounds a disc in the surface disjoint from the rest of the diagram.

Note that we do not call a simple closed curve which bounds a disc in the ambient surface a trivial doodle diagram unless the surface is a 2-sphere. For example, let (Σ, C)

be a doodle diagram such that Σ is a torus and C is a simple closed curve which bounds a disc in the torus. Then (Σ, C) is not a trivial doodle diagram in our sense. We call C a floating component, not a trivial diagram.

Lemma 3.5 *Let C be a floating component of a doodle diagram $S = (\Sigma, D)$ and let Σ_0 be the connected component of Σ containing C. If (Σ_0, C) is not a trivial doodle diagram, then there exists a doodle diagram S' such that $S \searrow S'$.*

Proof. We consider two cases:

(1) $(D \setminus C) \cap \Sigma_0 \neq \emptyset$, i.e., there exists a component of $D \setminus C$ on Σ_0.

(2) $(D \setminus C) \cap \Sigma_0 = \emptyset$, i.e., the rest of the diagram $D \setminus C$ misses Σ_0.

In case 1), apply an h^{-1} move along a simple closed curve surrounding C and we obtain an (admissible) doodle diagram which is the disjoint union of $(\Sigma, D \setminus C)$ and a trivial doodle diagram with one component.

In case 2), the surface Σ_0 has positive genus. We can apply an h^{-1} move along a simple closed curve on Σ_0 avoiding C to reduce the genus. $\square$

Proof. (Proof of Theorem 3.4) First we show that $\mathcal{D}$ has the FDPP property. Suppose that there is an infinite descending path and let $S = (\Sigma, D)$ be a vertex on the path. The number of crossings of D is an upper bound of the number of H_1^{-1} and H_2^{-1} moves appearing on the path. Since we are considering admissible doodle diagrams, the topology of Σ with the number of circles of D makes an bound of the numbers of $h^{\pm 1}$ moves appearing on the path. This contradicts to that the path is infinite. Thus $\mathcal{D}$ has the FDPP property.

We will show that $\mathcal{D}$ has the DC or LDC property by looking at the possible cases.

Suppose S_1, S_2, S_3 is a sequence of diagrams created by $H_1^{\pm 1}, H_2^{\pm 1}$ moves such that $S_1 \swarrow S_2 \searrow S_3$. The first move creates a region R and the second destroys a region R'. If $R = R'$ then the two moves cancel and S_1 is the same as S_3. The other cases are illustrated in Figure 4.

In case a) the regions R and R' are disjoint so by reversing the order of the moves there is an intermediate S_2' such that $S_1 \searrow S_2' \swarrow S_3$. The figure illustrates an H_2^+ and an H_1^-.

In case b) the regions R and R' have a point in common. The region R is created by an H_2^+ move and the region R' is created accidentally. The region R' is then deleted by an H_2^- move (subcase b-i, on the left of the figure) or by an H_1^- move (subcase b-ii, on the right of the figure). In case b-i) both moves can be dispensed with since S_1 is homeomorphic to S_3. In case b-ii) the result, S_3, could have been created with an H_1^+ move. So S_1 is joined to S_3 by an edge.

In case c) the regions R and R' have two points in common. In case c-i), on the left of the figure, let $S = (\Sigma, D)$ be a doodle diagram which is obtained from S_2 by removing the circle bounding $R \cup R'$. Then S_1 (or S_3, resp.) is obtained from S by adding a floating component C_1 (or C_3, resp.). Let S_2' be the disjoint union of S and a trivial doodle

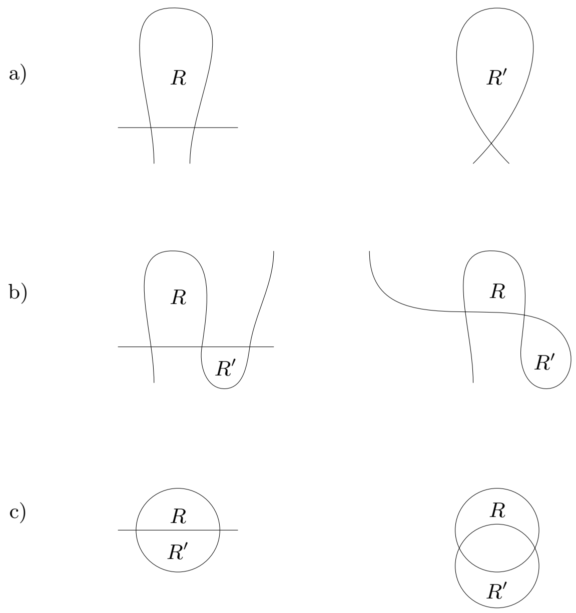

Figure 4: The regions R and R'

diagram with one component. Then, as in the case 1) of the proof of Lemma 3.5, we see that $S_1 \searrow S_2' \swarrow S_3$. In case c-ii), on the right of the figure, let $S_i = (\Sigma, D_i)$ $(i = 1, 2, 3)$ be the doodle diagrams, and let C_2 and C_2' be the circles of D_2 facing R and R' in the figure, respectively. In D_1 their corresponding circles C_1 and C_1' are lying such that C_1 is inside of C_1', and in D_3 their corresponding circles C_3 and C_3' are lying such that C_3' is inside of C_3. Note that $D_i = (D_2 \setminus (C_2 \cup C_2')) \cup (C_i \cup C_i')$ for $i = 1, 3$. Let Σ_0 be the component of the surface Σ which contains C_2 and C_2'. There are two subcases:

- c-ii-1) $(D_2 \setminus (C_2 \cup C_2')) \cap \Sigma_0 \neq \emptyset$.
- c-ii-2) $(D_2 \setminus (C_2 \cup C_2')) \cap \Sigma_0 = \emptyset$.

In case c-ii-1), let S_2' be the disjoint union of $(\Sigma, D_2 \setminus (C_2 \cup C_2'))$ and a trivial doodle diagram with two components. We see that there is a path $S_1 \searrow S_1' \searrow S_2' \swarrow S_3' \swarrow S_3$ for some S_1' and S_3' as in the case 1) of the proof of Lemma 3.5.

In case c-ii-2), let S_2' be the disjoint union of $(\Sigma \setminus \Sigma_0, D_2 \setminus (C_2 \cup C_2'))$ and a trivial doodle diagram with two components. By the argument of the proof of Lemma 3.5, we see that there is a path $S_1 \searrow S_1' \searrow S_2' \swarrow S_3' \swarrow S_3$ when Σ_0 is a 2-sphere, and there is a path $S_1 \searrow S_1' \searrow \cdots \searrow S_2' \swarrow \cdots \swarrow S_3' \swarrow S_3$ when Σ_0 has positive genus.

We now consider mixtures of $h^{\pm 1}$ and $H_1^{\pm 1}, H_2^{\pm 1}$ moves.

Suppose we have a simple peak $S_1 \swarrow S_2 \searrow S_3$. There are various cases to consider. Let $S_1 \swarrow S_2$ be an H_1^+ move and $S_2 \searrow S_3$ an h^- move. The expansion creates a monogon and the handle elimination takes place along an essential curve which is disjoint from the diagram and hence disjoint from the monogon. It follows that the two operations can be reversed.

A similar argument holds if the initial expansion creates a bigon with an H_2^+ move.

Now consider $S_1 \swarrow S_2 \searrow S_3$ in which the first expansion makes a handle from the boundary of disks D_1 and D_2 and the second move eliminates a bigon (or monogon). In either case this must be disjoint from the two disks in order to happen so the two moves can be reversed.

Finally, suppose the expansion, $S_1 \swarrow S_2$, creates a handle and the collapse, $S_2 \searrow S_3$, deletes a handle. This means that in the middle diagram S_2 there is a simple closed curve b_1 which is the belt of the added handle and a simple closed curve b_2 upon which the surgery takes place. Both curves are disjoint from the doodle diagram D_2 on Σ_2 with $S_2 = (\Sigma_2, D_2)$.

If b_1 and b_2 are disjoint then we can reverse the operations. If b_1 and b_2 are not disjoint then we can assume that they meet transversely. Let A_1 and A_2 be annular neighbourhoods of b_1 and b_2. If these are sufficiently thin then,

1. the components of $A_1 \cap A_2$ are square neighbourhoods of the intersection points of b_1 and b_2,
2. $A_1 \cup A_2$ is a connected surface with boundary,
3. $A_1 \cup A_2$ is disjoint from the doodle diagram D_2.

Let the components of $\partial(A_1 \cup A_2)$ be $c_1 \cup c_2 \cup \cdots \cup c_n$.

If each c_i is inessential then it spans a disc U_i in the surface Σ_2. So the (topological) components of the doodle diagram D_2 each lie in a disc. (It follows that the doodle is planar.) In this case, there is a sequence of h^- moves transforming S_1 into a doodle diagram $S' = (\Sigma', D')$ which is a disjoint union of doodle diagrams on 2-spheres, and there is a sequence of h^- moves transforming S_3 into the same $S' = (\Sigma', D')$.

Otherwise, at least one of the boundary components is essential: call it c. Then c is disjoint from b_1 and b_2.

Case 1) Suppose that c is a non-separating loop of Σ_2 or that c is a separating loop and each component of the surface obtained by the surgery on c intersects with the diagram D_2. Let S' be the result of surgery on c in S_2, which is an admissible doodle diagram. Let S'', (S''') be the result of surgery on b_1, (b_2) in S'. Note that S'', (S''') is also the result of surgery on c in S_1, (S_3).

So there is a path $S_1 \searrow S'' \swarrow S' \searrow S''' \swarrow S_3$ and an edge $S_2 \searrow S'$ which implies the local diamond condition.

It follows that the conditions of the proof reduction graph are satisfied and the theorem is proved.

Case 2) Suppose that c is a separating loop and one of the component of the surface obtained by the surgery on c does not intersect with D_2. In this case, the component missing D_2 is a closed connected surface with positive genus. This implies that we can first apply h^{-1} moves to S_1 and S_3 along simple closed curves on the surface killing the genus, and we can reduce the case into the case that c is inessential. $\square$

Definition 3.6 A doodle diagram is **minimal** if the interior of every region is simply connected and there are no monogons and bigons.

Theorem 3.4 implies a Kuperberg type theorem, see [18].

Theorem 3.7 *If two minimal diagrams represent the same oriented (or unoriented) doodle then they are homeomorphic.*

Proof. Since there are no monogons and bigons, no H_1^-, H_2^- moves can be initiated. Because the regions are simply connected the same is true for h^- moves. It follows that a minimal diagram represents a root and roots are unique. $\square$

A doodle is called **trivial** if it has a trivial doodle diagram as a representative.

By Theorem 3.7, a doodle is trivial if and only if its unique minimal diagram is a trivial doodle diagram.

The poppy and the Borromean doodle are non-trivial because the diagrams depicted in Figure 1 are minimal diagrams which are not trivial doodle diagrams. Similarly the Hopf doodle is non-trivial because the diagram on the torus with one square region is a minimal diagram which is not a trivial doodle diagram.

Theorem 3.4 also implies that a minimal diagram is characterized as a diagram with minimum crossing number and the ambient surface has minimum genus and maximum component number as described below.

For a diagram (Σ, D) let $c(D)$, $g(\Sigma)$ and $\mathrm{comp}(\Sigma)$ stand for the number of crossings of D, the genus of Σ and the number of connected components of Σ.

Theorem 3.8 *Let (Σ, D) be a doodle diagram. The following conditions are equivalent.*

(1) (Σ, D) *is a minimal diagram.*

(2) *For any diagram (Σ', D') representing the same doodle as (Σ, D), three inequalities $c(D) \leq c(D')$, $g(\Sigma) \leq g(\Sigma')$ and $\mathrm{comp}(\Sigma) \geq \mathrm{comp}(\Sigma')$ hold.*

Proof. Note that a doodle diagram is minimal if and only if it is a root of the graph $\mathcal{D}$ of doodle diagrams. (1) $\Rightarrow$ (2): Suppose (1). By Theorem 3.4, if (Σ, D) and (Σ', D') are not homeomorphic then there exists a descending path from (Σ', D') to (Σ, D). This implies the inequalities. (2) $\Rightarrow$ (1): It is obvious. $\square$

The **genus** of a doodle is the minimum genus among all surfaces on which the doodle has a representative. A doodle with genus 0 is a planar doodle. By Theorem 3.8 we have the following.

Corollary 3.9 *The genus of a doodle is the genus of the surface of a (unique) minimal diagram of the doodle.*

4 Examples of Minimal Doodle Diagrams

In this section we will look at minimal diagrams of doodles. Here we consider unoriented doodles.

4.1 Observation from Cell Decompositions of Surfaces

Let D be a minimal doodle diagram on a connected surface of genus g. Then the crossings, edges and regions form a cell decomposition of the ambient surface. Let V be the number of crossings, E the number of edges and F the number of regions. Let F_i be the number of i-gon regions, $i = 3, 4, \ldots$.

Lemma 4.1 *With the notation above,*

(I_1) $E = 2V$

(I_2) $F = V + 2 - 2g$

(I_3) $V - 6 + 6g = F_4 + 2F_5 + 3F_6 + \cdots$

Proof. Every vertex contributes 4 to the number of edges twice over. This proves identity I_1. The formula for the Euler-Poincaré number is $V - E + F = 2 - 2g$. Substituting I_1 gives I_2. Each i-gon contributes i edges to E. Each edge is the edge of 2 regions. So $4V = 2E = 3F_3 + 4F_4 + \cdots$ But $6 + 3V - 6g = 3F = 3F_3 + 3F_4 + \cdots$. Taking the difference implies I_3. □

Theorem 4.2 *Consider minimal doodle diagrams on the 2-sphere.*

(1) *They have at least 6 crossings.*

(2) *There is only one with 6 crossings: the Borromean doodle.*

(3) *There are none with 7 crossings.*

(4) *There is only one with 8 crossings: the poppy.*

Proof. (1) It follows immediately from identity I_3.

(2) If $V = 6$ then I_3 implies that there are only triangular regions. These form the octahedral decomposition of the 2-sphere.

(3) If $V = 7$ then I_3 implies that there exists a single tetragonal region and other regions are all triangular regions. Let $\{v_0, v_1, v_2, v_3\}$ be the vertices (crossings) of the tetragonal region, and $\{u_0, u_1, u_2\}$ the other three vertices. Note that two successive edges among the four edges v_iv_{i+1} ($i = 0, 1, 2, 3$) bounding the tetragonal region never bound a triangular region. Thus for each edge v_iv_{i+1} ($i = 0, 1, 2, 3$) of tetragonal region, there is a vertex w_i ($i = 0, 1, 2, 3$) $\in \{u_0, u_1, u_2\}$ such that $v_iv_{i+1}w_i$ is a triangular region. Without loss of generality, we may assume that $w_0 = w_2 = x$, $w_1 = a$ and $w_3 = b$ as in Figure 5, where the tetragonal region is outer-most. Two edges which connects to the crossing a (or b) are not depicted in this figure. It is impossible to draw such edges under this circumstance.

(4) I_3 implies that either (4-1) $F_4 = 2$ and $F_i = 0$ for $i > 4$ or (4-2) $F_5 = 1$ and $F_i = 0$ for $i = 4$ and for $i > 5$. By a similar argument with (3) we see that case (4-1) does not occur. In case (4-1), there are two tetragonal regions. If they are disjoint, then we have the poppy. If they share one or two vertices, we have a contradiction. □

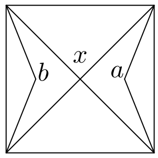

Figure 5: No Minimal Planar Diagram with 7 Crossings

4.2 Infinite sequences of planar doodles

Because we can recognize non-trivial doodles without 1-gons or 2-gons it is easy to invent sequences of different doodles. Here we define some sequences of planar doodles by giving sequences of minimal planar diagrams whose early members have other interpretations.

Example 4.3 There is a sequence of doodles $B_3, B_4, \ldots$ starting with the Borromean doodle, B_3 and the poppy, B_4. Let the vertices of the two concentric n-gons be $X_1X_2 \ldots X_n$ and $Y_1Y_2 \ldots Y_n$. Construct the squares $X_iX_{i+1}Y_{i+1}Y_i$, $i = 1, \ldots, n$ cyclically mod n. Join the diagonals X_i to Y_{i+1} sequentially to create the triangles. This defines B_n. It has $2n$ vertices, $2n$ triangular faces and 2 n-gon faces. The number of components is 3 if n is divisible by 3 and 1 otherwise. We call these doodles the **generalized Borromean** doodles.

Example 4.4 Another two sequences can be defined by taking 2 concentric n-gons separated by a concentric $2n$-gon and filling in the annular regions with alternate squares and triangles. This can be done in two ways so that at the $2n$-gon each square or triangle in one annulus abutts a single square or triangle in the other annulus. So taking nomenclature from the classification of polyhedra, we have **Gyro**, C'_n, which has the squares abutted by a common edge to the triangles whilst **Ortho**, C''_n, has the squares abutted by a common edge to the squares and the triangles abutted to the triangles. Both C'_n and C''_n have $4n$ vertices, $F_n = 2$ and $F_3 = F_4 = 2n$. The doodles C'_n and C''_n for $n > 3$ can be distinguished by the number of components. For $n = 3$ both C'_3 and C''_3 have 4 components and a similar number of triangular and square regions. They are illustrated in Figure 6 and can be distinguished by the combinatorics of their regions.

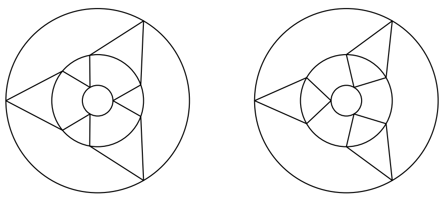

Figure 6: C'_3 and C''_3

We can describe C'_3 and C''_3 as follows. Consider the Borromean doodle, B_3, with circular components (see Figure 1). Now draw a circle separating the innermost triangular

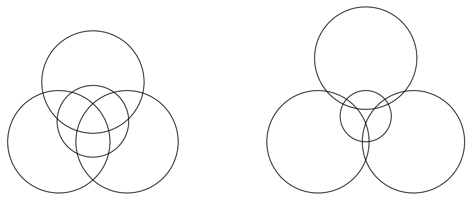

Figure 7: C_3' and C_3'' with Circular Components

region from the outermost triangular region. This describes C_3'. To obtain C_3'', perform an R_3 move on the innermost triangular region. See Figure 7.

If we remove any component of C_3' we get the Borromean doodle. On the other hand, if we remove the innermost circle from C_3'' we get the trivial doodle. This is another proof that C_3' and C_3'' are distinct.

Lemma 4.5 (1) *The doodle C_n' has four components if n is divisible by 3. Otherwise C_n' has two components.*

(2) *The doodle C_n'' has $n+1$ components.*

Proof. Firstly note that the central $2n$-gon is one of the components in both cases. (1) Let the vertices around one of the n-gons be $P_1 \dots P_n$. A component of C_n' containing the edge P_iP_{i+1}, contains the edge $P_{i+3}P_{i+4}$. (2) For C_n'' each component which isn't the central $2n$-gon is a hexagon containing the edge P_iP_{i+1}, $i = 1, \dots n \bmod n$, and there are n of these. □

Remark 4.6 (A Note on Planar Doodles and Polyhdra) There is a bijection between minimal planar doodles and the 1-skeleta of 3-dimensional polehdra whose vertices have valency four. It is well known that the Borromean doodle, B_3 is the 1-skeleton of the octahedron. In general B_n is the 1-skeleton of the n-gon antiprism, see [6] for definitions.

Furthermore C_3' is the 1-skeleton of the cuboctahedron and C_3'' is the 1-skeleton of the anticuboctahedron or triangular orthobicupola which is Johnson's solid J_{27}. C_4'/C_5' are also 1-skeleta of Johnson solids: J_{29}/J_{31}. $C_4''/\ C_5''$ are the 1-skeleta of the square gyrobicupola/pentagonal gyrobicupola and J_{28}/J_{30} are the 1-skeleta of the square orthobicupola/pentagonal orthobicupola. Further bicupola for $n > 5$ can not be realised with regular faces.

We are grateful to Peter Cromwell for this information.

4.3 The ± 1 construction

Let D be a doodle diagram on a surface Σ. Suppose that D has a region R with at least four edges and chose two disjoint edges e_1 and e_2 of R. Remove the interiors of the edges and join the dangling vertices with two diagonal arcs meeting at a new point X in the interior of R. This creates a new doodle diagram D'. We write $D \overset{+1}{\to} D'$ or $D' \overset{-1}{\to} D$ and call D an **ancestor** of D' and D' a **descendant** of D. The number of components of D' changes from that of D by 0 or ± 1, depending how e_1 and e_2 are oriented and placed.

Lemma 4.7 *Let D' have an ancestor D. Then D' is minimal if and only if D is minimal.*

Figure 8 illustrates how the poppy is the ancestor of (the unique) minimal diagram with 9 crossings and (the unique) two component minimal diagram with 10 crossings.

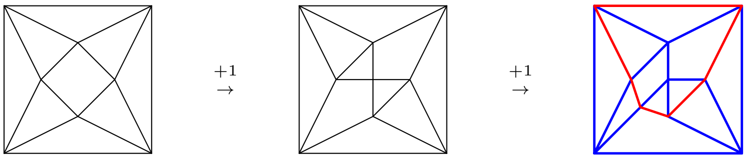

Figure 8: Descendants of the Poppy

Definition 4.8 A diagram is **fundamental** if it is a minimal diagram without ancestors.

The following lemma gives a procedure for recognising whether a diagram is fundamental or not.

Lemma 4.9 *If a minimal diagram has an ancestor, then there is a crossing X satisfying one of the following.*

(1) *There are two distinct regions appearing in a diagonal position about X such that they are a p'-gon and a q'-gon for some $p' > 3$ and $q' > 3$.*

(2) *There is a region appearing in a diagonal position about X such that it is an r'-gon for some $r' > 5$.*

Proof. Let D' be a descendant of D. Let e_1 , e_2 and R be the edges and the region of D that were used to produce D'. (1) Consider a case that the regions containing e_1 and e_2, beside R, are distinct. Suppose they are p-gon and q-gon. Since D is minimal, $p > 2$ and $q > 2$. After applying the $+1$ construction, these regions become a p'-region and a q'-region with $p' = p+1$ and $q' = q+1$. (2) Consider a case that the regions containing e_1 and e_2, beside R, are the same region of D. Suppose it is an r-gon. Since the boundary of this region contain e_1 and e_2, we have $r > 3$. By the $+1$ construction, this region becomes an r'-region with $r' = r + 2$. □

Theorem 4.10 *The generalised Borromean doodles are all fundamental.*

Proof. Their two regions with > 3 edges are protected by a ring of triangles. Thus there is no crossing satisfying the condition of the previous lemma. □

Figure 9: A (Flat) Crossing and a Virtual Crossing

5 Virtual Doodles

In this section we define a virtual doodle, which is an equivalence class of a virtual doodle diagram on the plane.

An **oriented** (or **unoriented**) **virtual doodle diagram** (on the plane) is a generically immersed oriented (or unoriented) circles on the plane such that some of the crossings are decorated by small circles. It is often regarded as an oriented 4-valent graph on the plane. When it is oriented, each crossing is as illustrated in Figure 9.

A crossing which is not encircled is called a **flat** crossing, a **real** crossing or sometimes simply a crossing. An encircled crossing is called a **virtual** crossing.

We consider moves on such diagrams.

We firstly consider moves R_1 and R_2 illustrated in Figure 10, where we should consider all possible orientations in the oriented case. They involve flat crossings, which are flat versions of the first Reidemeister move and the second. The shaded areas, a monogon and a bigon, have interiors disjoint from the rest of the diagram. We can divide these into two types: R_1^+ which creates the monogon and R_1^- which deletes it. Similarly we have R_2^+ and R_2^-.

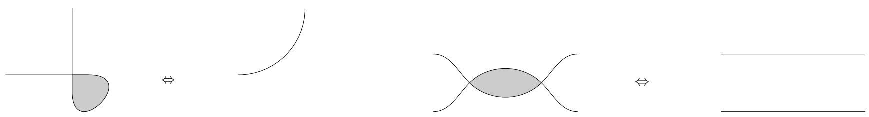

Figure 10: Moves R_1 (Left) and R_2 (Right)

Moves VR_1, VR_2 and VR_3 are illustrated in Figures 11 and 12(Left), which involve virtual crossings. As in the flat case, there are versions $VR_1^{\pm 1}$ and $VR_2^{\pm 1}$. For VR_3 in the oriented case, we can assume that all three arcs are oriented from left to right, (braid like), see [11].

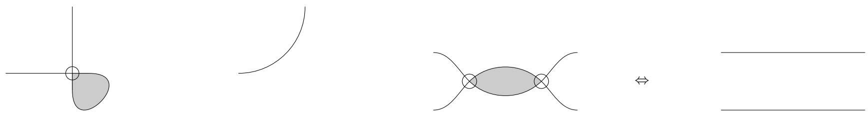

Figure 11: Moves VR_1 (Left) and VR_2 (Right)

Finally, we have a mixed move VR_4 illustrated in Figure 12(Right) in which two consecutive virtual crossings appear to move past a flat crossing.

Figure 12: Moves VR_3 (Left) and VR_4 (Right)

Definition 5.1 Two oriented (or unoriented) virtual doodle diagrams are **equivalent** if they are related by a sequence of the moves R_1, R_2, VR_1, VR_2, VR_3 and VR_4, modulo isotopies of the plane. An **oriented** (or **unoriented**) **virtual doodle** is an equivalence class of an oriented (or unoriented) virtual doodle diagram.

The move R_3 depicted in Figure 13(Left), which is the flat version of the third Reidemeister move, is forbidden in doodle theory. The move FVR_3 depicted in Figure 13(Right), in which two consecutive flat crossings appear to move past a virtual crossing is also forbidden.

Figure 13: Forbidden moves R_3 (Left) and FVR_3 (Right)

Remark 5.2 If we allow the forbidden move R_3 then the theory of doodles collapse to the theory of flat virtual knots and links. The flat Kishino knot diagram is depicted in Figure 14. It represents a non-trivial flat virtual knot, called the flat Kishino knot. (The original Kishino knot diagram is a diagram of a virtual knot. For a proof of the non-triviality as a virtual knot, see [3], [17].) The flat version was proved to be non-trivial in [13] and also in [8]. Thus the flat Kishino knot is also non-trivial as a flat doodle. Figure 15 illustrates the Kishino knot as a doodle representative on a genus-2 surface.

If we continue to forbid R_3 but allow the previously forbidden move FVR_3 then we get the theory of welded doodles. So far we have not been able to prove that non-trivial examples exist.

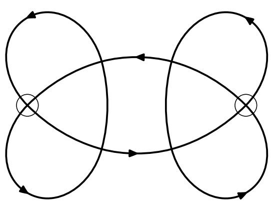

Figure 14: The Flat Kishino as a Flat Virtual Diagram

Call a sub-path of the diagram which only passes only through virtual crossings a **virtual path**. It is a consequence of VR_1, VR_2, VR_3 and VR_4 that a virtual path can

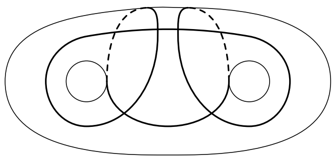

Figure 15: The Flat Kishino on a Surface of Genus 2

be moved any where in the diagram keeping its end points fixed provided that the new crossings engendered by this are virtual. Kauffman calls this the **detour** move, [16].

6 Virtual Doodles and Doodles Are the Same

We show that there is a bijection between oriented (or unoriented) virtual doodles (on the plane) and oriented (or unoriented) doodles (on surfaces). The idea is similar to that in [14] (and [5]) showing a bijection between virtual links and abstract links (or stable equivalence classes of link diagrams on surfaces).

In this section, we denote a virtual doodle diagram (on $\mathbb{R}^2$) by K and a doodle diagram (on a surface) by D.

Let K be a virtual doodle diagram with m flat crossings. Let $N_1, \ldots, N_m$ be regular neighbourhoods of the flat crossings and put $W := \mathrm{Cl}(\mathbb{R}^2 \setminus \cup_{i=1}^m N_i)$. The intersection $K \cap W$ is a union of arcs and loops immersed in W. (Each intersection point of $K \cap W$ is a virtual crossing of K. A loop appears when K has a component on which there are no flat crossings.) Let K' be another virtual doodle diagram with m flat crossings. Let σ be a bijection from the set of crossings of K to that of K'. Using an isotopy of $\mathbb{R}^2$, we assume that each crossing of K and the corresponding crossing of K' under σ are the same point of $\mathbb{R}^2$ and that K and K' are identical in $N_1, \ldots, N_m$. We say that K and K' have the same **Gauss data** with respect to σ if there is a bijection τ from the set of arcs and loops of $K \cap W$ to that of $K' \cap W$ such that for every arc e of $K \cap W$ the endpoints of e equal the endpoints of $\tau(e)$.

Lemma 6.1 *Let K and K' be virtual doodle diagrams with m flat crossings. The following conditions are equivalent:*

(1) *They have the same Gauss data with respect to a bijection between their flat crossings.*

(2) *They are related by a finite sequence of detour moves modulo isotopies of $\mathbb{R}^2$.*

(3) *They are related by a finite sequence of moves VR_1, VR_2, VR_3 and VR_4 modulo isotopies of $\mathbb{R}^2$.*

Proof. The equivalence between (1) and (2) is obvious. The equivalence between (2) and (3) is due to Kauffman [16] (cf. [14]). □

Let K be a virtual doodle diagram with m flat crossings. We continue with the definition of $N_1, \ldots, N_m$ and W as before.

Thickening the arcs and loops in W, we obtain bands and annuli immersed in W whose cores are $K \cap W$. The union of these bands and annuli with $N_1, \ldots, N_m$ is a compact oriented surface immersed in $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Here we assume the orientation of the surface is induced from the orientation of $\mathbb{R}^2$. Replacing it in neighbourhoods of virtual crossings as in Figure 16 in $\mathbb{R}^3$, we obtain a compact oriented surface, F, embedded in $\mathbb{R}^3$ and a diagram, D_F, with flat crossings on it. We denote the pair (F, D_F) by $\varphi(K)$. Here the pair (F, D_F) is considered up to orientation-preserving homeomorphic equivalence, and we ignore how it is embedded in $\mathbb{R}^3$.

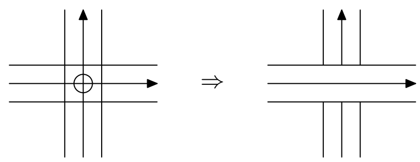

Figure 16: Topological Interpretation of a Virtual Crossing

Lemma 6.2 *If K' is obtained from K by VR_1, VR_2, VR_3 or VR_4, then $\varphi(K)$ is homeomorphic to $\varphi(K')$.*

Proof. By a move VR_1, VR_2, VR_3 or VR_4, $N_1, \ldots, N_m$ are preserved, and there is a bijection from the set of arcs and loops of $K \cap W$ to that of $K' \cap W$. It induces a homeomorphism from the union of the bands and annuli in $\varphi(K)$ and that in $\varphi(K')$. Sending $N_1, \ldots, N_m$ identically, we have a homeomorphism from $\varphi(K)$ to $\varphi(K')$. $\square$

Lemma 6.3 *If K' is obtained from K by $R_1^{\pm}$ or $R_2^{\pm}$, then there is a closed connected oriented surface Σ and doodle diagrams D and D' on Σ such that $(N(D, \Sigma), D)$ is homeomorphic to $\varphi(K)$, $(N(D', \Sigma), D')$ is homeomorphic to $\varphi(K')$ and D' is obtained from D by $H_1^{\pm}$ or $H_2^{\pm}$ on Σ.*

Proof. It suffices to consider the case where K' is obtained from K by R_1^- or R_2^-. Let $\varphi(K) = (F, D_F)$. Attaching 2-disks to F along the boundary, we have a closed oriented surface $\tilde{F}$ in which there is a monogon or a bigon where H_1^- or H_2^- can be applied. $\square$

So if K is a virtual doodle diagram and $\varphi(K) = (F, D_F)$ is as above then by attaching 2-disks, annuli, or even any compact connected oriented surfaces to F along the boundary, we have a closed oriented surface Σ and an (admissible) doodle diagram D such that $(N(D, \Sigma), D)$ is homeomorphic to $\varphi(K)$. We call it a **doodle diagram associated to** K. It is not unique as a representative, however it is unique as a doodle. We call it, as a doodle, the **doodle associated to** K.

Theorem 6.4 *We have a bijection Φ from the family of oriented (or unoriented) virtual doodles to the family of oriented (or unoriented) doodles by defining $\Phi([K])$ to be the doodle associated to K.*

Proof. We consider the oriented case. The unoriented case follows from the oriented case.

The well-definedness of Φ follows from the previous lemmas. We shall prove that Φ is surjective and then injective.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(x, y, z) \mapsto (x, y)$.

Let (Σ, D) be a doodle diagram D on a surface Σ. By an abuse of notation let $N_1, \ldots, N_m$ also stand for the regular neighbourhoods of crossings of D in Σ, and put $X := \mathrm{Cl}(\Sigma \setminus \cup_{i=1}^m N_i)$. The intersection $D \cap X$ is a union of arcs and loops embedded in X. The neighborhoods of the arcs and loops in X are bands and annuli in X, and the union of these bands and annuli together with $N_1, \ldots, N_m$ forms a regular neighborhood $N(D)$ of D in Σ. Consider an embedding $\tilde{g} : N(D) \to \mathbb{R}^3$ such that $\tilde{g}(\cup_{i=1}^m N_i) \subset \mathbb{R}^2 \times \{0\}$ and $g := \pi \circ \tilde{g} : N(D) \to \mathbb{R}^2$ is an orientation-preserving immersion whose singularity set is a union of transverse intersections among bands and annuli. Regard $g(D)$ as a virtual doodle diagram, say K, such that the flat crossings are in $g(\cup_{i=1}^m N_i)$ and the virtual crossings are in the intersections of bands and annuli. By definition, $(N(D), D)$ is $\varphi(K)$ and (Σ, D) is a doodle diagram associated to K. This shows that Φ is surjective. We call such a K a **virtual doodle diagram associated to** (Σ, D).

Before proving the injectivity of Φ, we introduce the notion of Gauss data for doodle diagrams on surfaces, which is analogous to the notion of Gauss data for virtual doodle diagrams.

Let (Σ, D) and (Σ', D') be doodle diagrams on surfaces with the same number of crossings. Let σ be a bijection from the set of crossings of D to that of D'. Let $N_1, \ldots, N_m$ be regular neighbourhoods of crossings of D and $N'_1, \ldots, N'_m$ be regular neighbourhoods of the corresponding crossings of D' under σ. Let $f : \cup_{i=1}^m N_i \to \cup_{i=1}^m N'_i$ be a homeomorphism such that for each i, $(N_i, D \cap N_i)$ is mapped to $(N'_i, D' \cap N'_i)$ homeomorphically with respect to their orientations. Let $X = \mathrm{Cl}(\Sigma \setminus \cup_{i=1}^m N_i)$ and $X' = \mathrm{Cl}(\Sigma' \setminus \cup_{i=1}^m N'_i)$. The intersection $D \cap X$ is a union of some arcs and loops embedded in X. We say that (Σ, D) and (Σ', D') have the same **Gauss data** with respect to σ if the homeomorphism $f : \cup_{i=1}^m N_i \to \cup_{i=1}^m N'_i$ has an extension to a homeomorphism from $\cup_{i=1}^m N_i \cup (D \cap X)$ to $\cup_{i=1}^m N'_i \cup (D' \cap X')$.

Note that doodle diagrams (Σ, D) and (Σ', D') have the same Gauss data with respect to a bijection between their crossings if and only if they are related by a finite sequence of homeomorphic equivalence and surface surgeries disjoint from diagrams.

Now we prove that Φ is injective. Let (Σ, D) and (Σ', D') be doodle diagrams on surfaces and let K and K' be their associated virtual doodle diagrams.

(1) Suppose that (Σ, D) and (Σ', D') have the same Gauss data with respect to a bijection between their crossings. Then K and K' have the same Gauss data with respect to a bijection between their flat crossings. By Lemma 6.1, we see that K and K' represent the same virtual doodle.

(2) Suppose that $\Sigma = \Sigma'$ and D' is obtained from D by a move H_1^- or H_2^-. In the construction of K and K', taking embeddings $\tilde{g} : N(D) \to \mathbb{R}^3$ and $\tilde{g}' : N(D') \to \mathbb{R}^3$ suitably, we can obtain $K = g(D)$ and $K' = g'(D')$ such that K' is obtained from K by a move R_1^- or R_2^-. Thus K and K' represent the same virtual doodle.

(3) Suppose that $\Sigma = \Sigma'$ and D' is obtained from D by a move H_1^+ or H_2^+. From the previous case, we see that K and K' represent the same virtual doodle.

Therefore we see that Φ is injective. □

For an example of a doodle illustrated in two ways see Figures 14 and 15.

7 Minimal Virtual Doodles and Genera

In this section we will look at minimal virtual doodle diagrams and discuss about the genera of doodles.

Definition 7.1 A virtual doodle diagram is **minimal** if a doodle diagram associated to it is minimal.

Note that a virtual doodle diagram K is minimal if and only if the doodle diagram obtained from $\varphi(K) = (F, D_F)$ by attaching discs along the boundary is a doodle diagram without monogons or bigons.

For a virtual doodle, a minimal virtual doodle diagram is not unique. However, we have the following.

Proposition 7.2 *Let K and K' be minimal oriented (or unoriented) virtual doodle diagrams representing the same oriented (or unoriented) virtual doodle. Then K and K' are related by a finite sequence of detour moves modulo isotopies of $\mathbb{R}^2$.*

Proof. Let (F, D) (or (F', D')) be the doodle diagram obtained from $\varphi(K)$ (or $\varphi(K')$) by attaching discs along the boundary. Since K and K' are minimal and represent the same virtual doodle, (F, D) and (F', D') are minimal and represent the same doodle. By Theorem 3.7, (F, D) and (F', D') are homeomorphic. This implies that K and K' have the same Gauss data. By Lemma 6.1, K and K' are related by a finite sequence of detour moves modulo isotopies of $\mathbb{R}^2$. $\square$

Example 7.3 Figure 17 shows a minimal unoriented doodle diagram with 3 crossings. We denote it by $d3.1$.

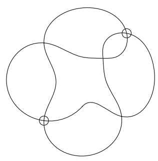

Figure 17: $d3.1$

Example 7.4 Figure 18 shows 19 minimal unoriented doodle diagrams with 4 crossings. They are not equivalent each other as unoriented virtual doodles.

Remark 7.5 The diagrams $d4.18$ and $d4.19$ in the previous version, arXiv:1612.08473v1, are equivalent as unoriented virtual doodles. The authors would like to thank Victoria Lebed for pointing out this.

Example 7.6 See Figure 19. It is a minimal unoriented virtual doodle diagram with 8 crossings. Since the associated minimal doodle diagram is on the torus, by Corollary 3.9 the genus of the doodle is 1. Hence we cannot remove a virtual crossing.

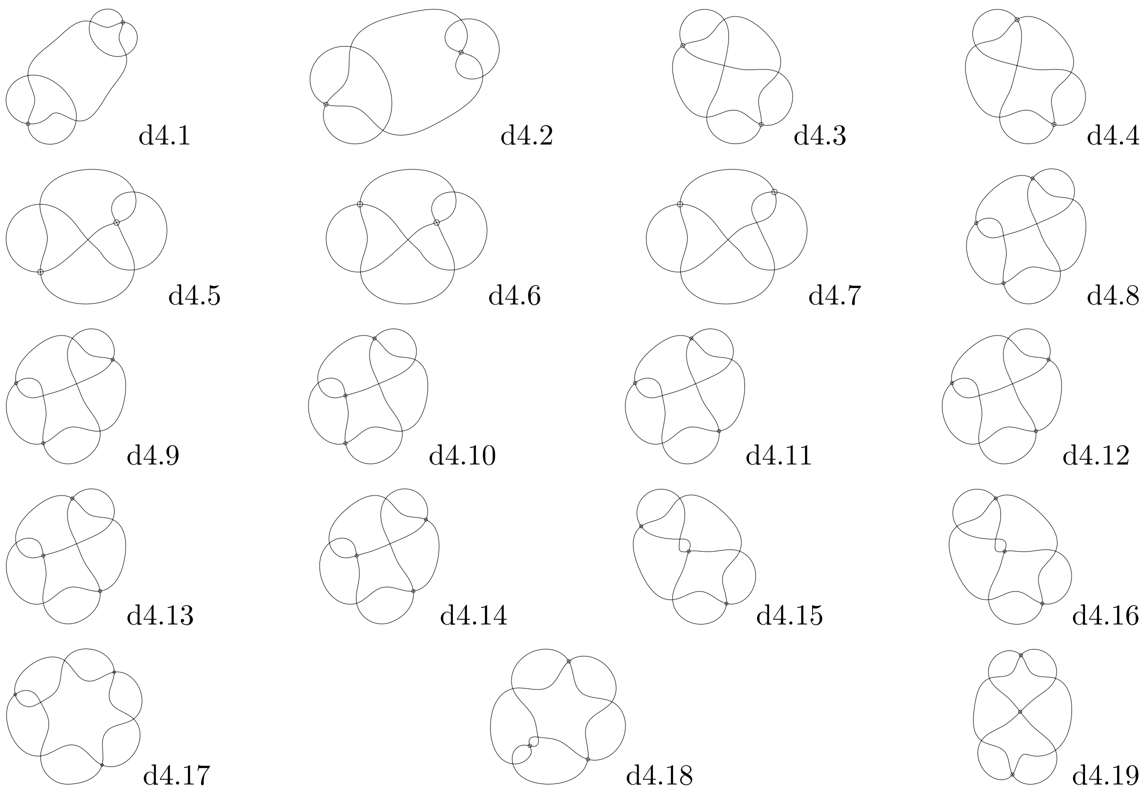

Figure 18: Minimal Diagrams with 4 Crossings

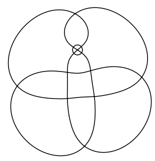

Figure 19: A Minimal Virtual Doodle Diagram with One Component and 8 Crossings

It is easily seen that if K has m virtual crossings then the genus of the doodle $\Phi([K])$ associated to K is equal to or less than m. Thus, the genus of the doodle is never greater than the minimum number of virtual crossings. It is quite easy to find examples where the inequality is strict.

Example 7.7 Let K be the virtual doodle diagram depicted in Figure 20. The doodle $\Phi([K])$ associated to K has a minimal diagram on the torus and hence the genus of $\Phi([K])$ is 1 by Corollary 3.9. On the other hand, the minimum number of virtual crossings among all virtual doodle diagrams equivalent to K is 2, since the left circle must have at least one virtual crossing with each of the circles on the right.

However it may be possible to amalgamate virtual crossings. If we take two consecutive virtual crossings on a virtual path then a surgery around the bases of the two handle

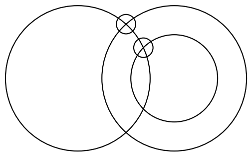

Figure 20: A Genus 1 Doodle with at Least 2 Virtual Crossings

representations means that the two handles can be replaced by one. If we apply this to the doodle in Figure 20 we can represent it on a torus.

Indeed we can generalize this as follows. Define a **virtual area** in a virtual diagram to be a square transverse to the diagram in which arcs enter in an edge and exit through the opposite edge with all the crossing points inside the square virtual. All these virtual crossings can be replaced by one handle. See Figure 21.

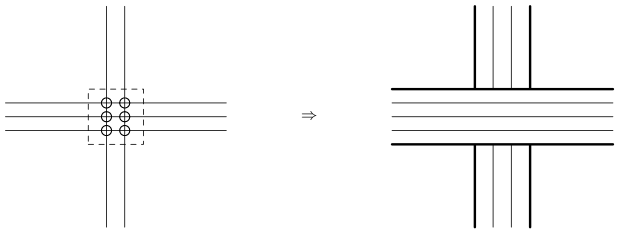

Figure 21: Topological Interpretation of a Virtual Area

Let K be a virtual doodle diagram. A collection of virtual areas of K, say $\mathcal{A} = \{A_1, \ldots, A_k\}$, is a **virtual area covering** of K if they cover all virtual crossings and if they are mutually disjoint. Let $\mathrm{va}(K)$ denote the minimum cardinal number of all virtual area coverings of K. When K has no virtual crossings, we assume $\mathrm{va}(K) = 0$. We call $\mathrm{va}(K)$ the **virtual area number** of K.

For the virtual doodle $[K]$ represented by K, we define $\mathrm{va}([K])$ by the minimum number among all $\mathrm{va}(K')$ such that K' is equivalent to K. We call it the **virtual area number** of the virtual doodle $[K]$.

Let Φ be the bijection from the family of virtual doodles to the family of doodles defined in Section 6.

Theorem 7.8 *Let K be a virtual doodle diagram and D a doodle such that $\Phi([K]) = [D]$. Then the virtual area number* $\mathrm{va}([K])$ *of the virtual doodle $[K]$ equals the genus of the doodle $[D]$.*

Proof. First we show that $g([D]) \leq \mathrm{va}([K])$. Let K_0 be a virtual doodle diagram with $[K_0] = [K]$ and $\mathrm{va}(K_0) = \mathrm{va}([K])$. Obviously, there is a doodle diagram D_0 on a closed surface F_0 with $[D_0] = \Phi([K_0])$ $(= \Phi([K]) = [D])$ and $g(F_0) = \mathrm{va}(K_0)$. Thus $g([D]) = g([D_0]) \leq g(F_0) = \mathrm{va}(K_0) = \mathrm{va}([K])$.

We show that $g([D]) \geq \mathrm{va}([K])$. Let D' be a doodle diagram on a closed surface F' with $[D'] = [D]$ and $g(F') = g([D])$. Let $g = g(F')$. Consider a standard handle decomposition of F', say $H^0 \cup H_1^1 \cup \ldots \cup H_{2g}^1 \cup H^2$, where we assume that the attaching areas of the 1-handles appear in the order

$$H_1^1, H_2^1, H_1^1, H_2^1, H_3^1, H_4^1, H_3^1, H_4^1, \ldots$$

on the boudary of H^0. By moving D' by an isotopy of F', we may assume that (1) all crossings of D' are in the interior of the 0-handle H^0, (2) for each 1-handle H_i^1 the intersection of $D' \cap H_i^1$ is empty or some arcs that are parallel copies of the core of the 1-handle, and (3) D' is disjoint from the 2-handle H^2. Consider an immmersion of $H^0 \cup H_1^1 \cup \ldots \cup H_{2g}^1$ to $\mathbb{R}^2$ such that the restriction of each handle is an embedding and the multiple point set consists of the transverse intersections of pairs of 1-handles H_{2i-1}^1 and H_{2i}^1 for $i = 1, \ldots, g$. Then we have a virtual doodle diagram, say K', such that $\Phi([K']) = [D']$ and K' has a virtual area coveing whose cardinal number is less than or equal to g. Thus, $\mathrm{va}([K']) \leq g$. Since $\Phi([K']) = [D'] = [D] = \Phi[K]$, we have $[K'] = [K]$. Thus $\mathrm{va}([K]) = \mathrm{va}([K']) \leq g = g(F') = g([D])$. $\square$

Remark 7.9 Clearly a result, similar to the above result, can be proved for virtual links.

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